the course makes with the wind after we lower sail. Let P be the turning point and G be the goal. Then, since the whole time of transit is the time that it would take to go the distance $OB + PG$ under motor, we must choose φ so that this distance shall be a minimum.

$$PG = \frac{(a \tan \theta - b) \cos \theta}{\sin (\theta - \varphi)}, \qquad OB = (a - PG \cos \varphi) \sec \beta.$$

$$OB + PG = a \sec \beta - PG(\cos \varphi \sec \beta - 1).$$

This will be a minimum when $PG \cos \varphi(\sec \beta - \sec \varphi)$ is a maximum; and that will be the case when u is a maximum if

$$u = \frac{\cos \theta \cos \varphi}{\sin (\theta - \varphi)} [\sec \beta - \sec \varphi] = \frac{\sec \beta - \sec \varphi}{\tan \theta - \tan \varphi}.$$

$$\frac{du}{d\varphi} = \frac{(\tan \varphi - \tan \theta) \sec \varphi \tan \varphi + (\sec \beta - \sec \varphi) \sec^2 \varphi}{(\tan \theta - \tan \varphi)^2} = 0.$$

$$(\tan \theta - \tan \varphi) \tan \varphi = (\sec \beta - \sec \varphi) \sec \varphi.$$

$$\tan \theta \tan \varphi + 1 = \sec \beta \sec \varphi.$$

$$\cos \varphi + \tan \theta \sin \varphi = \sec \beta. \tag{6}$$

φ determined from equation (6) gives the bearing of the goal when the proper turning point is reached.

If $\theta = 45°$ and $\beta = 18°$, $\varphi = 3°$. If $\theta = 41°.5$ and $\beta = 16°.5$, $\varphi = 3°-$.

If the yachtsman has patience and a soul that longs for accuracy he can amuse himself by trying a third approximation for $F(\theta)$, namely,

$$F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2 + c(\theta - \alpha)^3.$$

THE ACCELERATIONS OF THE POINTS OF A RIGID BODY.

By PETER FIELD AND ALEXANDER ZIWET, University of Michigan.

INTRODUCTION.

It has long been known that, in plane motion, the acceleration field is completely determined by the accelerations of any two points; and the point of zero acceleration, or "acceleration center," for such a motion is discussed in most works on mechanics.[1] The corresponding problems for three dimensions, that is, the determination of the acceleration field from the accelerations of three points and the construction of the acceleration center (or centers), are less widely known. Vector methods are particularly appropriate for their solution. They

[1] Compare AMERICAN MATHEMATICAL MONTHLY, Vol. XXI, 1914, pp. 105–113.

bring out very clearly to what extent the instantaneous motion remains indeterminate when the accelerations of three points are given, and lead to the enumeration of the various cases that may arise concerning the existence of points of zero acceleration. The dynamical interpretation of the result of Art. 20 would also appear to be noteworthy.

For the determination of the central axis from the accelerations of three points compare J. PETERSEN, *Kinematik,* 1884, Art. 36, pp. 47–49, and R. MEHMKE, in *Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt,* p. 77.

I. VELOCITY.

1. Let O, P, Q be any three non-collinear points of a rigid body and put

$$P - O = \boldsymbol{p}, \quad Q - O = \boldsymbol{q}$$

The assumption that O, P, Q are different and not collinear gives

$$\boldsymbol{p} \wedge \boldsymbol{q} \neq 0.$$

The assumption of rigidity gives

$$\boldsymbol{p}^2 = \text{const.}, \quad \boldsymbol{q}^2 = \text{const.}, \quad (\boldsymbol{q} - \boldsymbol{p})^2 = \text{const.}$$

Differentiating with respect to the time we find:

$$\dot{\boldsymbol{p}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \quad \dot{\boldsymbol{p}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{p} = 0, \tag{1}$$

where $\dot{\boldsymbol{p}}$, $\dot{\boldsymbol{q}}$ are the velocities of P, Q, *relative to* O.

If R be any fourth point of the body and we put $R - O = \boldsymbol{r}$ we have the additional rigidity conditions

$$\boldsymbol{r}^2 = \text{const.}, \quad (\boldsymbol{r} - \boldsymbol{p})^2 = \text{const.}, \quad (\boldsymbol{r} - \boldsymbol{q})^2 = \text{const.},$$

whence

$$\dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} + \dot{\boldsymbol{p}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{r} = 0. \tag{2}$$

We proceed to prove that *the velocities* $\dot{O}$, $\dot{P}$, $\dot{Q}$ *of any three non-collinear points* O, P, Q *of a rigid body completely determine the velocities of all points of the body.*

2. Suppose first that $\dot{\boldsymbol{p}} = 0$ and $\dot{\boldsymbol{q}} = 0$ so that O, P, Q have equal velocities. Then by (2)

$$\dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} = 0.$$

Now if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} \neq 0$ so that $\boldsymbol{r}$, $\boldsymbol{p}$, $\boldsymbol{q}$ can be taken as reference vectors for $\dot{\boldsymbol{r}}$ the equations show that $\dot{\boldsymbol{r}} = 0$. If, however, $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} = 0$ we can put $\boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q}$, whence $\dot{\boldsymbol{r}} = a\dot{\boldsymbol{p}} + b\dot{\boldsymbol{q}} = 0$, as before. Hence *if any three non-collinear points have equal velocities the velocities of all points are equal.* The instantaneous mot on in this case is called a **translation**.

3. Suppose next that $\dot{\boldsymbol{p}} = 0$, $\dot{\boldsymbol{q}} \neq 0$. The conditions (1) and (2) reduce to

$$\dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \quad \dot{\boldsymbol{q}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{r} = 0.$$

As by the first two conditions $\dot{\boldsymbol{q}}$ is normal to both $\boldsymbol{q}$ and $\boldsymbol{p}$ we can put $\dot{\boldsymbol{q}} = k\boldsymbol{p} \wedge \boldsymbol{q}$. The last equation then becomes $\dot{\boldsymbol{r}} \times \boldsymbol{q} + k\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{r} = 0$, *i. e.*, $(\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{q} = 0$; the last three equations can therefore be written in the form

$$(\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{r} = 0, \qquad (\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{p} = 0, \qquad (\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{q} = 0;$$

hence, if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} \neq 0$:

$$\dot{\boldsymbol{r}} = k\boldsymbol{p} \wedge \boldsymbol{r};$$

if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} = 0$ we can again put $\boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q}$, whence $\dot{\boldsymbol{r}} = b\dot{\boldsymbol{q}} = bk\boldsymbol{p} \wedge \boldsymbol{q} = k\boldsymbol{p} \wedge (\boldsymbol{r} - a\boldsymbol{p}) = k\boldsymbol{p} \wedge \boldsymbol{r}$. Thus, if $\dot{O} = \dot{P} \neq \dot{Q}$, there exists a vector $\boldsymbol{u} = k\boldsymbol{p}$, parallel to OP, such that the velocity of every point R can be derived from it by the formula

$$\dot{\boldsymbol{r}} = \boldsymbol{u} \wedge \boldsymbol{r}.$$

The scalar k is determined by observing that $\dot{\boldsymbol{q}} = k\boldsymbol{p} \wedge \boldsymbol{q}$ whence

$$k = \frac{\operatorname{mod} \dot{\boldsymbol{q}}}{\operatorname{mod} \boldsymbol{p} \cdot \operatorname{mod} \boldsymbol{q} \cdot \sin(\boldsymbol{p}, \boldsymbol{q})}.$$

4. Finally, suppose that $\dot{\boldsymbol{p}} \neq 0$ and $\dot{\boldsymbol{q}} \neq 0$. It can be shown that in this case, too, there exists a vector $\boldsymbol{w}$ such that

$$\dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r},$$

whatever the point R. For, as the vector $\boldsymbol{w}$ is to satisfy the conditions

$$\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}, \qquad \dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q},$$

it must be normal to both $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$; it must therefore be of the form

$$\boldsymbol{w} = k\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}.$$

To determine k we have only to substitute this value of $\boldsymbol{w}$ in the two preceding equations. Owing to the first two of (1) we find

$$\dot{\boldsymbol{p}} = k(\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}) \wedge \boldsymbol{p} = -k\dot{\boldsymbol{q}} \times \boldsymbol{p} \cdot \dot{\boldsymbol{p}},$$

$$\dot{\boldsymbol{q}} = k(\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}) \wedge \boldsymbol{q} = \quad k\dot{\boldsymbol{p}} \times \boldsymbol{q} \cdot \dot{\boldsymbol{q}};$$

and owing to the third of (1) these equations give the same value for k, viz.,

$$k = -\frac{1}{\dot{\boldsymbol{q}} \times \boldsymbol{p}} = \frac{1}{\dot{\boldsymbol{p}} \times \boldsymbol{q}}.$$

The vector

(3) $$\boldsymbol{w} = -\frac{\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}}{\dot{\boldsymbol{q}} \times \boldsymbol{p}} = \frac{\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}}{\dot{\boldsymbol{p}} \times \boldsymbol{q}}$$

gives therefore $\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q}$. That it also gives

(4) $$\dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r},$$

whatever R, appears by putting $R - O = \boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q}$, whence

$$\begin{aligned}\dot{\boldsymbol{r}} &= a\dot{\boldsymbol{p}} + b\dot{\boldsymbol{q}} + c(\dot{\boldsymbol{p}} \wedge \boldsymbol{q} + \boldsymbol{p} \wedge \dot{\boldsymbol{q}})\\ &= a\boldsymbol{w} \wedge \boldsymbol{p} + b\boldsymbol{w} \wedge \boldsymbol{q} + c[(\boldsymbol{w} \wedge \boldsymbol{p}) \wedge \boldsymbol{q} + \boldsymbol{p} \wedge (\boldsymbol{w} \wedge \boldsymbol{q})]\\ &= a\boldsymbol{w} \wedge \boldsymbol{p} + b\boldsymbol{w} \wedge \boldsymbol{q} + c\boldsymbol{w} \wedge (\boldsymbol{p} \wedge \boldsymbol{q})\\ &= \boldsymbol{w} \wedge (a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q}) = \boldsymbol{w} \wedge \boldsymbol{r}.\end{aligned}$$

It is thus proved that *the velocities of all points of the body can be found from those of any three non-collinear points.*

5. The velocities $\dot{R}$, $\dot{S}$ of two points R, S are equal if $\dot{\boldsymbol{r}} = \dot{\boldsymbol{s}}$, *i. e.*, if $\boldsymbol{w} \wedge \boldsymbol{r} = \boldsymbol{w} \wedge \boldsymbol{s}$, whence $\boldsymbol{w} \wedge (\boldsymbol{r} - \boldsymbol{s}) = 0$. It follows that all points of any line parallel to $\boldsymbol{w}$ have equal velocities. This shows that the case of Art. 3, where two of the given velocities were assumed equal, is no less general than that of Art. 4; but the case of Art. 3 suggests more directly the introduction of the auxiliary vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

In general, there is no point of zero velocity. But if such a point exists, say O, then all points of the line l through O, parallel to $\boldsymbol{w}$, have zero velocity, and the velocity $\dot{R} = \dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r}$ of every other point R is normal to $\boldsymbol{w}$ and $\boldsymbol{r}$ and in magnitude proportional to the distance of R from l. The instantaneous motion in this case is called a **rotation**; the line l is called the **axis**, the vector $\boldsymbol{w}$ the **angular velocity**, of the rotation.

6. Even in the general case the vector $\boldsymbol{w}$ may be called the angular velocity. The instantaneous motion is completely determined by the angular velocity $\boldsymbol{w}$ and the linear velocity $\dot{O}$ of any one point. For then the velocity of every point R is, by (4):

$$\dot{R} = \dot{O} + \boldsymbol{w} \wedge \boldsymbol{r}.$$

II. Acceleration.

7. Differentiating the equations $\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q}$ we find:

$$(5)\qquad \begin{aligned}\ddot{\boldsymbol{p}} &= \dot{\boldsymbol{w}} \wedge \boldsymbol{p} + \boldsymbol{w} \wedge \dot{\boldsymbol{p}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{p} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{p}),\\ \ddot{\boldsymbol{q}} &= \dot{\boldsymbol{w}} \wedge \boldsymbol{q} + \boldsymbol{w} \wedge \dot{\boldsymbol{q}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{q} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q}).\end{aligned}$$

These equations are not linear in $\boldsymbol{w}$. Eliminating $\dot{\boldsymbol{w}}$ we have:

$$(5')\qquad \begin{aligned}\ddot{\boldsymbol{p}} \times \boldsymbol{p} &= \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{p}) \times \boldsymbol{p} = (\boldsymbol{w} \times \boldsymbol{p})^2 - \boldsymbol{w}^2\boldsymbol{p}^2,\\ \ddot{\boldsymbol{q}} \times \boldsymbol{q} &= \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q}) \times \boldsymbol{q} = (\boldsymbol{w} \times \boldsymbol{q})^2 - \boldsymbol{w}^2\boldsymbol{q}^2,\\ \ddot{\boldsymbol{p}} \times \boldsymbol{q} + \ddot{\boldsymbol{q}} \times \boldsymbol{p} &= 2(\boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{p} \times \boldsymbol{q} \cdot \boldsymbol{w}^2).\end{aligned}$$

If in the trihedral formed by $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{w}$ at O (Fig. 1) we denote the face angles $(\boldsymbol{p}, \boldsymbol{q})$, $(\boldsymbol{p}, \boldsymbol{w})$, $(\boldsymbol{q}, \boldsymbol{w})$ by a, b, c and the dihedral angle at $\boldsymbol{w}$ by α, and if we put

for the sake of brevity

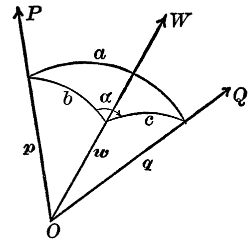

Fig. 1.

$$-\frac{\ddot{\boldsymbol{p}} \times \boldsymbol{p}}{\boldsymbol{p}^2} = B, \qquad -\frac{\ddot{\boldsymbol{q}} \times \boldsymbol{q}}{\boldsymbol{q}^2} = C,$$

$$-\frac{\ddot{\boldsymbol{p}} \times \boldsymbol{q} + \ddot{\boldsymbol{q}} \times \boldsymbol{p}}{2 \operatorname{mod} \boldsymbol{p} \cdot \operatorname{mod} \boldsymbol{q}} = A,$$

the equations become

(6) $\boldsymbol{w}^2 \sin^2 b = B, \quad \boldsymbol{w}^2 \sin^2 c = C, \quad \boldsymbol{w}^2 \sin b \sin c \cos \alpha = A,$

whence

$$\cos \alpha = \frac{A}{\sqrt{BC}}. \tag{7}$$

When the accelerations of the three points O, P, Q are given, the quantities A, B, C can be found. But the vector $\boldsymbol{w}$ is not uniquely determined: its extremity W (Fig. 1) may lie on either side of the plane OPQ, and the sense of $\boldsymbol{w}$ remains indeterminate. Thus there are four distinct solutions.

Moreover, the accelerations $\ddot{O}, \ddot{P}, \ddot{Q}$ cannot be arbitrarily prescribed. To obtain a real $\boldsymbol{w}$ it is necessary that $A^2 \leqq BC$, *i. e.*,

$$(\ddot{\boldsymbol{p}} \times \boldsymbol{q} + \ddot{\boldsymbol{q}} \times \boldsymbol{p})^2 \leqq 4\ddot{\boldsymbol{p}} \times \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} \times \boldsymbol{q}.$$

8. Suppose, in particular, that the points O, P, Q are such that $\boldsymbol{p}$ and $\boldsymbol{q}$ are rectangular unit vectors and put

$$\boldsymbol{w} = w_1 \boldsymbol{p} + w_2 \boldsymbol{q} + w_3 \boldsymbol{p} \wedge \boldsymbol{q},$$

so that w_1, w_2, w_3 are the rectangular coordinates of $\boldsymbol{w}$ with respect to $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p} \wedge \boldsymbol{q}$. If, moreover, p_1, p_2, p_3 and q_1, q_2, q_3 are the coordinates of $\ddot{\boldsymbol{p}}$ and $\ddot{\boldsymbol{q}}$ with respect to the same reference set, the equations (5′) become

$$w_2^2 + w_3^2 = -p_1, \quad w_3^2 + w_1^2 = -q_2, \quad 2w_1w_2 = p_2 + q_1. \tag{8}$$

These equations give

$$\begin{aligned} w_1^2 &= \quad \tfrac{1}{2}[p_1 - q_2 + \sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2}], \\ w_2^2 &= -\tfrac{1}{2}[p_1 - q_2 - \sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2}], \end{aligned} \tag{9}$$

where the sign of the square root has been selected so as to make w_1 and w_2 real. For w_3 we find:

$$w_3^2 = -p_1 - w_2^2 = -q_2 - w_1^2 = -\tfrac{1}{2}[p_1 + q_2 + \sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2}].$$

Hence w_3 is real only if

$$p_1 q_2 \geqq \left(\frac{p_2 + q_1}{2}\right)^2,$$

which is the condition at the end of Art. 7.

9. *To express $\dot{\boldsymbol{w}}$ in terms of $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$, and $\boldsymbol{w}$* we may either differentiate (3) or observe that, referring $\dot{\boldsymbol{w}}$ to $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{w}$, we must have:

$$\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \dot{\boldsymbol{w}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \boldsymbol{p} + \boldsymbol{p} \wedge \dot{\boldsymbol{w}} \times \boldsymbol{w} \cdot \boldsymbol{q} + \boldsymbol{p} \wedge \boldsymbol{q} \times \dot{\boldsymbol{w}} \cdot \boldsymbol{w}$$

Now by (5)

$$\dot{\boldsymbol{w}} \wedge \boldsymbol{q} \times \boldsymbol{w} = \ddot{\boldsymbol{q}} \times \boldsymbol{w}, \qquad \boldsymbol{p} \wedge \dot{\boldsymbol{w}} \times \boldsymbol{w} = -\ddot{\boldsymbol{p}} \times \boldsymbol{w},$$

$$\begin{aligned} \boldsymbol{p} \wedge \boldsymbol{q} \times \dot{\boldsymbol{w}} = \boldsymbol{p} \times \boldsymbol{q} \wedge \dot{\boldsymbol{w}} &= -\boldsymbol{p} \times \ddot{\boldsymbol{q}} + \boldsymbol{p} \times \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q}) \\ &= -\boldsymbol{p} \times \ddot{\boldsymbol{q}} + \boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{w}^2 \cdot \boldsymbol{p} \times \boldsymbol{q}; \end{aligned}$$

hence

$$\begin{aligned} \boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \dot{\boldsymbol{w}} &= \boldsymbol{w} \times \ddot{\boldsymbol{q}} \cdot \boldsymbol{p} - \boldsymbol{w} \times \ddot{\boldsymbol{p}} \cdot \boldsymbol{q} \\ &\quad + (\boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{w}^2 \cdot \boldsymbol{p} \times \boldsymbol{q} - \boldsymbol{p} \times \ddot{\boldsymbol{q}})\boldsymbol{w}. \end{aligned} \tag{10}$$

The vector $\dot{\boldsymbol{w}}$ may be called the *angular acceleration.*

10. *Acceleration of any point R.* Putting, as in Art. 4,

$$R - O = \boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q},$$

we have

$$\ddot{\boldsymbol{r}} = a\ddot{\boldsymbol{p}} + b\ddot{\boldsymbol{q}} + c(\ddot{\boldsymbol{p}} \wedge \boldsymbol{q} + \boldsymbol{p} \wedge \ddot{\boldsymbol{q}} + 2\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}).$$

It may be observed that the acceleration of every point of the plane OPQ is given by

$$\ddot{\boldsymbol{r}} = a\ddot{\boldsymbol{p}} + b\ddot{\boldsymbol{q}};$$

i. e., it is determined completely by the accelerations $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$, of any three non-collinear points of this plane. But the acceleration of any point R not in this plane depends in addition, on the vector $\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}} = \dot{\boldsymbol{p}} \times \boldsymbol{q} \cdot \boldsymbol{w}$.

On the other hand, the relation (4) gives

$$\ddot{\boldsymbol{r}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}); \tag{11}$$

i. e., the acceleration $\ddot{R}$ of every point R is determined by the vectors $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$ and the acceleration $\ddot{O}$ of any one point O.

III. Center of Acceleration.

11. To see whether a point R of zero acceleration (or a *center of acceleration*) exists, let us first assume that $\boldsymbol{w} \times \dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \neq 0$, *i. e.*, $(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 \neq 0$, so that we can put

$$R - O = \boldsymbol{r} = a\boldsymbol{w} + b\dot{\boldsymbol{w}} + c\boldsymbol{w} \wedge \dot{\boldsymbol{w}}.$$

We then have to determine a, b, c so as to make $\ddot{R} = 0$, *i. e.*, by (11),

$$\ddot{O} + \dot{\boldsymbol{w}} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = 0.$$

Substituting for $\boldsymbol{r}$ its value we obtain the condition:

$$\ddot{O} + a\dot{\boldsymbol{w}} \wedge \boldsymbol{w} + c\dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) + b\boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) - c\boldsymbol{w}^2 \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0.$$

Multiplying by $\times \boldsymbol{w}$, $\times \dot{\boldsymbol{w}}$, $\times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})$ we find for a, b, c the conditions:

$$\ddot{O} \times \boldsymbol{w} + c\dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \times \boldsymbol{w} = 0,$$

$$\ddot{O} \times \dot{\boldsymbol{w}} + b\boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \times \dot{\boldsymbol{w}} = 0,$$

$$\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} - a(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 - c\boldsymbol{w}^2 \cdot (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 = 0,$$

whence

$$a = \frac{\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w}^2}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}, \qquad b = \frac{\ddot{O} \times \dot{\boldsymbol{w}}}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}, \qquad c = -\frac{\ddot{O} \times \boldsymbol{w}}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}.$$

The acceleration center is therefore given by the equation

$$(12) \quad (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2\boldsymbol{r} = \ddot{O} \times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \boldsymbol{w}^2 \cdot \boldsymbol{w}) \cdot \boldsymbol{w} + \ddot{O} \times \dot{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}.$$

This equation shows that whenever $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} \neq 0$ there exists one and only one acceleration center; if in particular $\ddot{O} = 0$, O is the center.

12. It remains to discuss the cases when $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. We first assume $\ddot{O} \neq 0$.

(*a*) Suppose $\ddot{O} \neq 0$, $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. We then have $\dot{\boldsymbol{w}} = k\boldsymbol{w}$, where $k \neq 0$, and the condition for R to have zero acceleration becomes

$$\ddot{O} + k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = 0.$$

Multiplying by $\boldsymbol{w} \times$ we find $\boldsymbol{w} \times \ddot{O} = 0$ as a first necessary condition for the existence of an acceleration center in this case.

It follows that $\boldsymbol{w} \wedge \ddot{O} \neq 0$ so that we can put $\boldsymbol{r} = a\boldsymbol{w} + b\ddot{O} + c\boldsymbol{w} \wedge \ddot{O}$; substituting this value we find if $\boldsymbol{w} \times \ddot{O} = 0$:

$$[1 - (kc + b)\boldsymbol{w}^2]\ddot{O} + (bk - c\boldsymbol{w}^2)\boldsymbol{w} \wedge \ddot{O} = 0,$$

whence

$$(kc + b)\boldsymbol{w}^2 = 1, \qquad kb = c\boldsymbol{w}^2,$$

i. e.,

$$b = \frac{1}{k^2 + \boldsymbol{w}^2}, \qquad c = \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)},$$

while a remains indeterminate. Thus, *if* $\ddot{O} \neq 0$, $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, *while* $\dot{\boldsymbol{w}}$ *is parallel to* $\boldsymbol{w}$, *there is no point of zero acceleration unless* $\ddot{O}$ *is normal to* $\boldsymbol{w}$; *if* $\dot{\boldsymbol{w}} \times \ddot{O} = 0$, *every point of the line*

$$R = O + a\boldsymbol{w} + \frac{1}{k^2 + \boldsymbol{w}^2}\ddot{O} + \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)}\boldsymbol{w} \wedge \ddot{O},$$

and no other point, has zero acceleration.

(*b*) *Under the same conditions, except that* $\dot{\boldsymbol{w}} = 0$, *there is no acceleration center if* $\boldsymbol{w} \times \ddot{O} \neq 0$; *if* $\boldsymbol{w} \times \ddot{O} = 0$, *the points of the line*

$$R = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\ddot{O}$$

have zero acceleration.

(c) If $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$ we have $\ddot{R} = \ddot{O} + \dot{\boldsymbol{w}} \wedge \boldsymbol{r}$; hence a first condition is $\dot{\boldsymbol{w}} \times \ddot{O} = 0$. We can therefore put $\boldsymbol{r} = a\dot{\boldsymbol{w}} + b\ddot{O} + c\dot{\boldsymbol{w}} \wedge \ddot{O}$ so that

$$\ddot{R} = \ddot{O} + b\dot{\boldsymbol{w}} \wedge \ddot{O} - c\dot{\boldsymbol{w}}^2 \cdot \ddot{O} = 0;$$

hence

$$b = 0, \qquad c = \frac{1}{\dot{\boldsymbol{w}}^2},$$

while a is arbitrary. Thus, *if* $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$ *there is no acceleration center if* $\dot{\boldsymbol{w}} \times \ddot{O} \neq 0$; *if* $\dot{\boldsymbol{w}}$ *is normal to* $\ddot{O}$ *all points of the line*

$$R = O + a\dot{\boldsymbol{w}} + \frac{1}{\dot{\boldsymbol{w}}^2}\dot{\boldsymbol{w}} \wedge \ddot{O}$$

have zero acceleration.

(d) *If* $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} = 0$ *there is evidently no point of zero acceleration.*

13. Next suppose $\ddot{O} = 0$ and $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. Then we have the following cases:

(e) $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$. As $\dot{\boldsymbol{w}} = k\boldsymbol{w}$, where $k \neq 0$, the condition for R to have zero acceleration is

$$0 = k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \times \boldsymbol{r} \cdot \boldsymbol{w} - \boldsymbol{w}^2 \cdot \boldsymbol{r}.$$

Hence in this case *every point of the line through* O *parallel to* $\boldsymbol{w}$ *has zero acceleration.*

(f) $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} = 0$. The result is evidently the same.

(g) $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$. As $\ddot{R} = \dot{\boldsymbol{w}} \wedge \boldsymbol{r}$, *every point of the line through* O *parallel to* $\dot{\boldsymbol{w}}$ *has zero acceleration.*

(h) $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} = 0$. As $\ddot{R} = 0$, *every point has zero acceleration.*

IV. Applications.

14. *Example I: Projectile whose axis of spin is permanent.* Let O be the centroid; as $\dot{\boldsymbol{w}} = 0$ we have case (b) of Art. 12. Hence there is no point of zero acceleration unless $\boldsymbol{w} \times \ddot{O} = 0$.

Suppose this condition satisfied. Let O_0 be the initial position of the centroid; $\boldsymbol{j}_0$, $\boldsymbol{k}_0$ unit vectors, $\boldsymbol{j}_0$ horizontal and at right angles to $\boldsymbol{w}$, $\boldsymbol{k}_0$ vertical downwards. Then, if v_1, v_2, v_3 are the coördinates of the velocity of O along $\boldsymbol{w}$, $\boldsymbol{j}_0$, $\boldsymbol{k}_0$, the position of the centroid at the time t is $O = O_0 + v_1 t\boldsymbol{w} + v_2 t\boldsymbol{j}_0 + (v_3 t + \frac{1}{2}gt^2)\boldsymbol{k}_0$; hence

$$R = O_0 + v_1 t\boldsymbol{w} + v_2 t\boldsymbol{j}_0 + (v_3 t + \tfrac{1}{2}gt^2)\boldsymbol{k}_0 + a\boldsymbol{w} + \frac{g}{\boldsymbol{w}^2}\boldsymbol{k}_0.$$

This is, for a constant t, the line whose points have zero accelerations. For t as a variable parameter, the same equation represents the ruled surface whose generators are those lines of space which in the course of time come to have this property. Thus, the locus in space of all those points that have zero acceleration is a parabolic cylinder.

To obtain the locus, in the body, of these lines, let $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors fixed

in the body and initially coinciding with $\boldsymbol{j}_0$, $\boldsymbol{k}_0$. Then we have for any point S of the body:

$$S = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}(\boldsymbol{g} \times \boldsymbol{j} \cdot \boldsymbol{j} + \boldsymbol{g} \times \boldsymbol{k} \cdot \boldsymbol{k})$$

$$= O + a\boldsymbol{w} + \frac{g}{\boldsymbol{w}^2}\sin wt \cdot \boldsymbol{j} + \frac{g}{\boldsymbol{w}^2}\cos wt \cdot \boldsymbol{k},$$

where $g = \text{mod}\, \boldsymbol{g}$, $w = \text{mod}\, \boldsymbol{w}$. The surface is a circular cylinder.

15. *Example II: Homogeneous circular disk rolling down an inclined plane, starting from rest.*

Let O be the position of the centroid at the time t. We have $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$; *i. e.*, we have case (*a*) of Art. 12, with $\boldsymbol{w} \times \ddot{O} = 0$. Hence, if O_0 is the initial position of the centroid, the points of zero acceleration are given by the equation

$$R = O_0 + (O - O_0) + a\boldsymbol{w} + \frac{1}{k^2 + \boldsymbol{w}^2}\ddot{O} + \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)}\boldsymbol{w} \wedge \ddot{O}.$$

If we put

$$\ddot{O} = a_1\boldsymbol{u}_0, \quad O - O_0 = \tfrac{1}{2}a_1t^2\boldsymbol{u}_0, \quad \boldsymbol{w} = a_2t\boldsymbol{w}_0, \quad k = \frac{\text{mod}\,\dot{\boldsymbol{w}}}{\text{mod}\,\boldsymbol{w}} = \frac{1}{t}, \quad \boldsymbol{w}_0 \wedge \boldsymbol{u}_0 = \boldsymbol{v}_0,$$

the equation becomes

$$R = O_0 + \tfrac{1}{2}a_1t^2\boldsymbol{u}_0 + aa_2t\boldsymbol{w}_0 + \frac{t^2}{1 + a_2{}^2t^4}a_1\boldsymbol{u}_0 + \frac{a_1}{a_2(1 + a_2{}^2t^4)}\boldsymbol{v}_0$$

$$= O_0 + aa_2t\boldsymbol{w}_0 + \frac{3 + a_2t^4}{2(1 + a_2{}^2t^4)}a_1t^2\boldsymbol{u}_0 + \frac{a_1}{a_2(1 + a_2{}^2t^4)}\boldsymbol{v}_0.$$

This is the locus, in space, of the points of zero acceleration. To find the locus in the body let $\boldsymbol{u}$, $\boldsymbol{v}$ be unit vectors fixed in the body and initially coinciding with $\boldsymbol{u}_0$, $\boldsymbol{v}_0$. The angle made by $\boldsymbol{u}$ with $\boldsymbol{u}_0$ is $\frac{1}{2}a_2t^2$, and the equation of the locus is

$$S = O + aa_2t\boldsymbol{w}_0 + \frac{t^2}{1 + a_2t^4}(a_1\boldsymbol{u}_0 \times \boldsymbol{u} \cdot \boldsymbol{u} + a_1\boldsymbol{u}_0 \times \boldsymbol{v} \cdot \boldsymbol{v})$$

$$+ \frac{a_1}{a_2(1 + a_2{}^2t^4)}(\boldsymbol{v}_0 \times \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v}_0 \times \boldsymbol{v} \cdot \boldsymbol{v}),$$

where

$$\boldsymbol{u}_0 \times \boldsymbol{u} = \cos\tfrac{1}{2}a_2t^2, \qquad \boldsymbol{u}_0 \times \boldsymbol{v} = -\sin\tfrac{1}{2}a_2t^2, \qquad \boldsymbol{v}_0 \times \boldsymbol{u} = \sin\tfrac{1}{2}a_2t^2,$$

$$\boldsymbol{v}_0 \times \boldsymbol{v} = \cos\tfrac{1}{2}a_2t^2.$$

16. *Example III: Projectile whose momental ellipsoid at the centroid is an ellipsoid of revolution.*

This is the general case so that we have to use equation (12), viz.,

$$R = O + \frac{1}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}[(\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \boldsymbol{w}^2 \cdot \ddot{O} \times \boldsymbol{w})\boldsymbol{w} + \ddot{O} \times \dot{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}].$$

Taking for O the centroid (Fig. 2) we have $\ddot{O} = \boldsymbol{g}$, $\ddot{O} \times \dot{\boldsymbol{w}} = 0$, and $\boldsymbol{w} \times \dot{\boldsymbol{w}} = 0$; hence, denoting the moduli of $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$, $\boldsymbol{g}$ by w, $\dot{w}$, g, and the constant angle made by $\boldsymbol{w}$ with $\boldsymbol{g}$ by θ, we have:

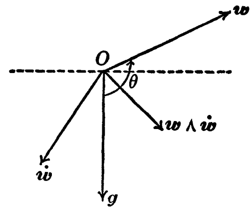

Fig. 2.

$$R = O + \frac{1}{\boldsymbol{w}^2\dot{\boldsymbol{w}}^2}[(gw\dot{w} \sin\theta + gw^3 \cos\theta)\boldsymbol{w} - gw\cos\theta \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}].$$

This shows that the body locus is a circle while the space locus is a transcendental curve.

Central Axis.

17. At any given instant t, the locus of those points of the body whose velocity is parallel to $\boldsymbol{w}$ is called the *instantaneous screw axis* of the motion, or the *central axis* of the velocity field. Assuming $\boldsymbol{w} \neq 0$, *i. e.*, assuming that the instantaneous motion is not a translation, it is readily shown that this locus is a straight line parallel to $\boldsymbol{w}$.

For, if the velocity $\dot{R} = \dot{O} + \boldsymbol{w} \wedge \boldsymbol{r}$ of any point R of the body is to be parallel to $\boldsymbol{w}$ we must have $\dot{R} \wedge \boldsymbol{w} = 0$, *i. e.*,

$$\dot{O} \wedge \boldsymbol{w} + (\boldsymbol{w} \wedge \boldsymbol{r}) \wedge \boldsymbol{w} = 0.$$

Putting $\boldsymbol{r} = a\boldsymbol{w} + b\dot{O} + c\boldsymbol{w} \wedge \dot{O}$ we find

$$\dot{O} \wedge \boldsymbol{w} + [\boldsymbol{w} \wedge (b\dot{O} + c\boldsymbol{w} \wedge \dot{O})] \wedge \boldsymbol{w} = 0,$$

i. e.,

$$\dot{O} \wedge \boldsymbol{w} + b(\boldsymbol{w} \wedge \dot{O}) \wedge \boldsymbol{w} + c\boldsymbol{w}^2 \cdot \boldsymbol{w} \wedge \dot{O} = 0.$$

Hence $b = 0$, $c = 1/\boldsymbol{w}^2$, while a remains arbitrary. The equation of the central axis at the time t is therefore

$$R = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}. \tag{13}$$

Every value of a gives a point R of the central axis; if R_0 corresponds to $a = 0$, we have $R - R_0 = a\boldsymbol{w}$; *i. e.*, the central axis is parallel to $\boldsymbol{w}$.

18. If in (13) the point O and the vectors $\boldsymbol{w}$ and $\dot{O}$ are given as functions of the time t, so that R becomes a function of the parameters a and t, this equation (13) represents the ruled surface formed by those lines of the body which in the course of time become central axes.

To obtain the equation of the ruled surface formed by those lines of space

which in the course of time become central axes, let O_1 be a point fixed in space and $\dot{O}$ the velocity of that point of the body which at the time t coincides with O_1. Then the required equation is

$$R = O_1 + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}. \tag{14}$$

19. By (14), the velocity of any point R of the central axis is

$$\dot{R} = a\dot{\boldsymbol{w}} + \frac{d}{dt}\left(\frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}\right).$$

The ruled surface (14) is developable, *i. e.*, the motion of the central axis is a pure rotation, if the velocity $\dot{R}$ is parallel to the plane of the vectors $\boldsymbol{w}$ and $\dot{\boldsymbol{w}}$, *i. e.*, if $\dot{R} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. The condition for the surface (14) to be developable is therefore

$$\left(\frac{d}{dt}\frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}\right) \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0,$$

i. e.,

$$\left(-\frac{2\boldsymbol{w} \times \dot{\boldsymbol{w}}}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O} + \dot{\boldsymbol{w}} \wedge \dot{O} + \boldsymbol{w} \wedge \ddot{O}\right) \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0. \tag{15}$$

20. This equation contains $\ddot{O}$ only in the combination $\ddot{O} \times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \wedge \boldsymbol{w}$; *i. e.*, if $\ddot{O}$ be resolved along the rectangular vectors $\boldsymbol{w}$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}}$, and $\boldsymbol{u} = \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})$, the equation contains only the component of $\ddot{O}$ along $\boldsymbol{u}$.

This fact has an interesting dynamical interpretation. Let O be the centroid, and let $\boldsymbol{R}$, $\boldsymbol{H}$ be the vectors of the resultant force and couple for O. Then our result means that, if the surface of the screw axes is to be developable, $\boldsymbol{R}$ must have a fixed projection on $\boldsymbol{u}$; in other words, all values of $\boldsymbol{R}$ that give a developable surface are such that, when drawn from O, their extremities lie in a plane parallel to $\boldsymbol{w}$ and $\boldsymbol{w} \wedge \dot{\boldsymbol{w}}$.

BOOK REVIEWS.

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Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated and provided with an introduction and notes by PHILIP E. B. JOURDAIN. The Open Court Publishing Company, Chicago, 1915. ix+211 pages.

The main purpose of this little volume, the first in the Open Court Series of Classics of Science and Philosophy, is to make accessible in English the two final papers of Georg Cantor on the theory of transfinite numbers, which embody the culmination of his researches on this subject, and which appeared in 1895 and 1897 in the *Mathematische Annalen* (Vols. 46 and 49). In order to give a proper setting to these two papers, the translator precedes them by an introduction of about 80 pages, in which he sketches briefly the progress of ideas which lead up to